XX. *On the Theory of the Elliptic Transcendents.* By JAMES IVORY, *A.M. F.R.S. Instit. Reg. Sc. Paris, et Soc. Reg. Sc. Gottin., Corresp.*

Read June 9, 1831.

THE branch of the integral calculus which treats of elliptic transcendents originated in the researches of FAGNANI, an Italian geometer of eminence. He discovered that two arcs of the periphery of a given ellipse may be determined in many ways, so that their difference shall be equal to an assignable straight line; and he proved that any arc of the lemniscata, like that of a circle, may be multiplied any number of times, or may be subdivided into any number of equal parts, by finite algebraic equations. These are particular results; and it was the discoveries of EULER that enabled geometers to advance to the investigation of the general properties of the elliptic functions. An integral in finite terms deduced by that geometer from an equation between the differentials of two similar transcendent quantities not separately integrable, led immediately to an algebraic equation between the amplitudes of three elliptic functions, of which one is the sum, or the difference, of the other two. This sort of integrals, therefore, could now be added or subtracted in a manner analogous to circular arcs, or logarithms; the amplitude of the sum, or of the difference, being expressed algebraically by means of the amplitudes of the quantities added or subtracted. What FAGNANI had accomplished with respect to the arcs of the lemniscata, which are expressed by a particular elliptic integral, EULER extended to all transcendents of the same class. To multiply a function of this kind, or to subdivide it into equal parts, was reduced to solving an algebraic equation. In general, all the properties of the elliptic transcendents, in which the modulus remains unchanged, are deducible from the discoveries of EULER. LANDEN enlarged our knowledge of this kind of functions, and made a useful addition to analysis, by showing that the arcs of the hyperbola may be reduced, by a proper transformation, to those of the ellipse. Every part of analysis is indebted to LAGRANGE, who enriched this particular branch with a general method for changing an elliptic function into another having a different modulus, a process which greatly facilitates the numerical calculation of this class of integrals. An elliptic function lies between an arc of the circle on one hand, and a logarithm on the other, approaching indefinitely to the first when the modulus is diminished to zero, and to the second when the modulus is augmented to unit, its other limit. By repeatedly applying the transformation of LAGRANGE, we may compute either a scale of decreasing moduli reducing the integral to a circular arc, or a scale of increasing moduli bringing it continually nearer to a logarithm. The approximation is very elegant and simple, and attains the end proposed with great rapidity.

The discoveries that have been mentioned occurred in the general cultivation of analysis; but LEGENDRE has bestowed much of his attention and study upon this particular branch of the integral calculus. He distributed the elliptic functions in distinct classes, and reduced them to a regular theory. In a Mémoire sur les Transcendantes Elliptiques, published in 1793, and in his Exercices de Calcul Intégral, which appeared in 1817, he has developed many of their properties entirely new; investigated the easiest methods of approximating to their values; computed numerical tables to facilitate their application; and exemplified their use in some interesting problems of geometry and mechanics. In a publication so late as 1825, the author, returning to the same subject, has rendered his theory still more perfect, and made many additions to it which further researches had suggested. In particular we find a new method of making an elliptic function approach as near as we please to a circular arc, or to a logarithm, by a scale of reduction very different from that of which LAGRANGE is the author, the only one before known. This step in advance would unavoidably have conducted to a more extensive theory of this kind of integrals, which, nearly about the same time, was being discovered by the researches of other geometers.

M. ABEL of Christiana, and M. JACOBI of Konigsberg, entirely changed the aspect of this branch of analysis by the extent and importance of their discoveries. The first of these geometers, whom, to the great loss of science, a premature death cut off in the beginning of a career of the highest expectations, happily conceived the idea of expressing the amplitude of an elliptic function in terms of the function itself. By this procedure the sines and cosines of the amplitudes become periodical quantities like the sines and cosines of circular arcs; and analogy immediately points out many new and useful properties which it would be difficult to deduce by any other mode of investigation. This new way of considering the subject struck out by M. ABEL, not only disclosed to him some interesting and original views, but it conducted him to the general and recondite theorems which, without his knowledge, had been previously discovered by the geometer of Konigsberg. M. JACOBI, following in his researches a different method from M. ABEL, proved that an elliptic function may be transformed innumerable ways into another similar function to which it bears constantly the same proportion. In the solution of this problem the modulus and the amplitude sought are deduced from the like given quantities, by equations which depend upon the division into an odd number of equal parts of the definite integral, having its amplitude equal to 90°; and, as any odd number may be chosen at pleasure, the number of transformations is unlimited. In consequence of this discovery, an elliptic function can have its modulus augmented or diminished according to an infinite number of different scales. The new process for effecting the same reduction discovered by LEGENDRE in 1825, is only the most simple case of the extensive theorem of M. JACOBI; and, although the older transformation of LAGRANGE is no part of the same theorem, it bears to it a close resemblance in every respect. Such is the principal addition made to this branch of analysis by M. JACOBI; but the new methods of investigation introduced by him and M. ABEL, open a wide field of collateral research, which probably will long continue to furnish matter for exercising the ingenuity of mathematicians.

But it seldom happens that an inventor arrives by the shortest road at the results which he has created, or explains them in the simplest manner. The demonstrations of M. JACOBI require long and complicated calculations; and it can hardly be said that the train of deduction leads naturally to the truths which are proved, or presents all the conclusions which the theory embraces in a connected point of view. The theorem does not comprehend the transformation of LAGRANGE, which must be separately demonstrated. This is an imperfection of no great moment; but it is always satisfactory to contemplate a theory in its full extent, and to deduce all the connected truths from the same principles. On a careful examination it will be found that the sines or cosines of the amplitudes used in the transformations are analogous to the sines or cosines of two circular arcs, one of which is a multiple of the other; insomuch that the former quantities are changed into the latter when the modulus is supposed to vanish in the algebraic expressions. We may therefore transfer to the elliptic transcendent the same methods of investigation that succeed in the circle. When this procedure is followed, there is no need to distinguish between an odd and an even number; the demonstrations are shortened; and the difficulties are mostly removed by the close analogy between the two cases. It is in this point of view that the subject is treated in this paper, in which it is proposed to demonstrate the principal theorems without going into the detail of the applications.

1. Elliptic functions of the first kind are of this form *, viz.

$$\int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

$$\int_0^{\psi} \frac{d\psi}{\sqrt{1 - h^2 \sin^2 \psi}}:$$

the arcs φ and ψ being the amplitudes, and the quantities k and h, which are always less than unit, the moduli of the functions. For the sake of abridging, I shall denote the foregoing integrals by K (φ) and H (ψ), the prefixes K and H having reference to the moduli k and h; and, for the definite integral between the amplitudes 0 and $\frac{\pi}{2}$, I shall use indiscriminately either K $\left(\frac{\pi}{2}\right)$ and H $\left(\frac{\pi}{2}\right)$, or, more simply, K and H.

The general equation to be investigated is the following,

$$\int \frac{d\psi}{\sqrt{1 - h^2 \sin^2 \psi}} = \beta \int \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \qquad (1)$$

β being a constant quantity equal to the first ratio of the nascent arcs ψ and φ.

* In what follows, the terms 'elliptic functions' and 'elliptic transcendents' are to be understood as applying to those of the first kind only, which alone are treated of.

If we admit that this is a possible equation, and suppose that when ψ is successively equal to the arcs of the series,

$$0, \quad \frac{\pi}{2}, \quad 2\frac{\pi}{2}, \quad 3\frac{\pi}{2}, \text{ \&c.},$$

φ attains the respective values,

$$0, \lambda_1, \lambda_2, \lambda_3, \text{ \&c.};$$

we shall have,

$$\mathrm{H} = \beta\,\mathrm{K}\,(\lambda_1),\ 2\,\mathrm{H} = \beta\,\mathrm{K}\,(\lambda_2),\ 3\,\mathrm{H} = \beta\,\mathrm{K}\,(\lambda_3), \text{ \&c.};$$

and consequently,

$$\mathrm{K}\,(\lambda_2) = 2\,\mathrm{K}\,(\lambda_1), \quad \mathrm{K}\,(\lambda_3) = 3\,\mathrm{K}\,(\lambda_1), \text{ \&c.}$$

Thus the arcs λ_2, λ_3, &c. are the amplitudes of the multiples of the function $\mathrm{K}\ (\lambda_1)$, which itself remains indeterminate. We may therefore suppose $p \times \mathrm{K}\,(\lambda_1) = \mathrm{K}\left(\frac{\pi}{2}\right)$, p representing any integer number; and, in consequence, we shall have

$$\mathrm{K}\,(\lambda_1) = \frac{1}{p}\,\mathrm{K}, \quad \mathrm{K}\,(\lambda_2) = \frac{2}{p}\,\mathrm{K}, \quad \ldots\ \mathrm{K}\,(\lambda_m) = \frac{m}{p}\,\mathrm{K}.$$

Any proposed number being assumed for p, we may determine the amplitudes λ_1, λ_2, λ_3, &c. by the theory for the multiplication and subdivision of elliptic functions: but as the equations to be solved are complicated and impracticable, the arcs λ_1, λ_2, &c. may be treated as known quantities without any attempt to compute them.

An elliptic function becomes equal to the arc of its amplitude, when the modulus vanishes: and in this case the arcs λ_1, λ_2, λ_3, &c. are obtained by the subdivision of the quadrant of the circle, and are respectively equal to $\frac{1}{p} \cdot \frac{\pi}{2}$, $\frac{2}{p} \cdot \frac{\pi}{2}$, $\frac{3}{p} \cdot \frac{\pi}{2}$, &c.

Having made these observations, we shall for the present dismiss all consideration of the equation to be demonstrated, and turn our attention to investigate two variable arcs ψ and φ, such that the first shall have the successive values,

$$0, \quad \frac{\pi}{2}, \quad 2 \times \frac{\pi}{2}, \quad 3 \times \frac{\pi}{2}, \text{ \&c.}$$

when the second becomes respectively equal to the several known amplitudes,

$$0, \lambda_1, \lambda_2, \lambda_3, \&c.$$

2. As we shall have occasion to refer to the formulas for the addition and subtraction of elliptic functions, it will be convenient to premise them.

Let a and b represent any two amplitudes, and put

$$\mathrm{K}(a) + \mathrm{K}(b) = \mathrm{K}(s)$$
$$\mathrm{K}(a) - \mathrm{K}(b) = \mathrm{K}(\sigma):$$

then, according to the formulas of EULER *,

$$\sin s = \frac{\sin a \cos b \sqrt{1 - k^2 \sin^2 b} + \cos a \sin b \sqrt{1 - k^2 \sin^2 a}}{1 - k^2 \sin^2 a \sin^2 b}$$

$$\sin \sigma = \frac{\sin a \cos b \sqrt{1 - k^2 \sin^2 b} - \cos a \sin b \sqrt{1 - k^2 \sin^2 a}}{1 - k^2 \sin^2 a \sin^2 b}.$$

From these we immediately deduce,

$$\sin s \sin \sigma = \frac{\sin^2 a - \sin^2 b}{1 - k^2 \sin^2 a \sin^2 b} \dagger. \qquad \text{(A)}$$

It may be observed that if $a = \lambda_m$, $b = \lambda_n$; then $s = \lambda_{m+n}$, $\sigma = \lambda_{m-n}$: for it is obvious that

$$\mathrm{K}(\lambda_m) + \mathrm{K}(\lambda_n) = (m + n)\,\mathrm{K}(\lambda_1) = \mathrm{K}(\lambda_{m+n})$$
$$\mathrm{K}(\lambda_m) - \mathrm{K}(\lambda_n) = (m - n)\,\mathrm{K}(\lambda_1) = \mathrm{K}(\lambda_{m-n})$$

3. In order to avail ourselves of the analogy between the elliptic functions and the arcs of a circle, we must take that view of the matter first suggested by M. ABEL. Let

$$u = \int_0^\varphi \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = \mathrm{K}(\varphi);$$

then, as u is a variable quantity depending upon the amplitude φ, reciprocally this latter quantity will depend upon the first; which dependance we shall express in this manner,

$$\varphi = \text{amplitude of } u = \mathrm{A}\,u,$$
$$\sin \varphi = \sin \mathrm{A}\,u.$$

* Traité des Fonctions Elliptiques, tom. i. p. 22.

† This equation is called by M. ABEL "la propriété fondamentale."

For the sake of abridging, let $\omega = \frac{1}{p}$ K; so that λ_1, λ_2, λ_3, &c. will be the respective amplitudes of ω, $2\,\omega$, $3\,\omega$, &c.; and λ_p = amplitude of $p\,\omega$ = amp. of K = 90°; and λ_{2p} = amp. of $2\,p\,\omega$ = amp. of 2 K = 180°. From the nature of the integral, it follows that when u receives an addition equal to $2\,p\,\omega$ or 2 K, the amplitude of u will be increased by 180°.

To the indeterminate quantity u let there be added the several even multiples of ω less than $2\,p\,\omega$; and let us consider the sines of the amplitudes of the functions so formed, viz.

$$\sin A\, u,\ \sin A\,(u + 2\,\omega),\ \sin A\,(u + 4\,\omega),\ \ldots\ \sin A\,(u + 2\,p\,\omega - 2\,\omega):$$

in this series, if we substitute in place of u, the successive quantities $u + 2\,\omega$, $u + 4\,\omega$, $u + 6\,\omega$, &c., the same sines will constantly recur in periodical order, abstracting from the change of sign when an amplitude becomes greater than 180°, or than a multiple of 180°. Thus, if we put $u + 2\,\omega$ in place of u, the second term of the foregoing series will stand first, and the last term will be $\sin A\,(u + 2\,p\,\omega) = -\sin A\,(u)$. In like manner, if $u + 4\,\omega$ be substituted for u, the third term of the series will stand first, and the two last terms will be $-\sin A\, u$, $-\sin A\,(u + 2\,\omega)$; and so on.

Let us now put

$$\gamma = \sin A\,\omega \times \sin A\,(3\,\omega) \times \sin A\,(5\,\omega)\ \ldots \times \sin A\,(2\,p\,\omega - \omega)$$

or, which is the same thing,

$$\gamma = \sin \lambda_1 \times \sin \lambda_3 \times \sin \lambda_5\ \ldots \times \sin \lambda_{2p-1};$$

and further, let us assume,

$$y = \frac{\sin A\, u \times \sin A\,(u + 2\,\omega) \times \sin A\,(u + 4\,\omega) \ldots \times \sin A\,(u + 2\,p\,\omega - 2\,\omega)}{\gamma}. \quad \text{(B)}$$

In this expression, if we substitute for u, the several odd multiples of ω in succession, viz.

ω, $3\,\omega$, $5\,\omega$, $7\,\omega$, &c.

it follows, from what has been said, that the products in the numerator will always be the same, and equal to the denominator, but that their signs will change alternately as the successive quantities are substituted. Thus, when any odd multiple $(2n+1)\omega$ is substituted for u in the expression (B), the value of y is always $+1$ or -1, according as $(2n+1)\omega$ holds an odd or an even rank in the series of the odd multiples of ω.

On the other hand, when u is zero, or equal to $2n\omega$ any even multiple of ω, we shall have $y=0$, one of the factors of the numerator necessarily vanishing; for in a sequence of the even multiples of ω, of which the number is p, there must be one equal to $2p\omega$, or to a multiple of $2p\omega$; and therefore when $u=2n\omega$, one of the factors must be the sine of an amplitude equal to 180° or to a multiple of 180°.

Further, let $\omega - z$ be substituted for u in the expression (B), z being less than ω; then,

$$y = \frac{\sin A(\omega - z) \sin A(3\omega - z) \dots \sin A(2p\omega - \omega - z)}{\gamma}.$$

Now, in the numerator, the partial products, of the first and last factors, of the second and last but one, and so on, are as follows:

$$\sin A(\omega - z) \sin A(2p\omega - \omega - z) = \sin A(\omega - z) \sin A(\omega + z),$$

$$\sin A(3\omega - z) \sin A(2p\omega - 3\omega - z) = \sin A(3\omega - z) \sin A(3\omega + z), \text{\&c.}$$

to which we must add the single factor $\sin A(p\omega - z)$, when p is an odd number. All the partial products, it will be observed, have the same value whether z be positive or negative; and they are all greatest, when $z=0$, as will readily appear from what is proved in § 2. Wherefore y has the same value and the same sign, when u is at equal distances from the limits 0 and 2ω; and it attains its greatest magnitude, equal to 1, when $u=\omega$. And, if we substitute $(2n+1)\omega - z$ for u, this substitution will not change the foregoing factors, but only their order, and the sign of their product, which sign, while u is contained between the limits $2n\omega$ and $2n\omega + 2\omega$, will be $+$ or $-$, according as $(2n+1)\omega$ holds an odd or an even rank in the series of the odd multiples of ω.

We may now conclude, from what has been proved, that y, in the expression (B), represents the sine of an arc ψ, which increases from zero with the elliptic function u, and coincides with the successive terms of the series,

$$0, \ \frac{\pi}{2}, \ 2\frac{\pi}{2}, \ 3\frac{\pi}{2}, \text{ \&c. ad infinitum,}$$

at the same time that u attains the values,

$0, \omega, 2\omega, 3\omega$, &c. ad infinitum,

or, when the amplitude of u becomes equal to the several known arcs,

$0, \lambda_1, \lambda_2, \lambda_3$, &c. ad infinitum:

and further, that there is but one value of y, or of $\sin \psi$, between the two consecutive terms $m \times \frac{\pi}{2}$ and $(m+1) \times \frac{\pi}{2}$, for any given value of u between the limits $m\omega$ and $(m+1)\omega$, or for any given amplitude between the arcs λ_m and λ_{m+1}.

4. In what has been proved, p may be either an odd or an even number; but we must now distinguish between the two cases, in like manner as it is necessary to do when we investigate the sine of a multiple of a circular arc. Representing the amplitude of u by ϕ, we shall have, $u = \mathrm{K}(\phi)$, and $\sin\phi = \sin \mathrm{A} u$. When p is odd, there will be an even number of factors after the first in the numerator of the expression of y or $\sin\psi$; and any one of these, as $\sin \mathrm{A}(u + 2n\omega)$, will have another, namely, $\sin \mathrm{A}(u + 2p\omega - 2n\omega) = \sin \mathrm{A}(2n\omega - u)$, answering to it; and the product of this pair of factors, viz. $\sin \mathrm{A}(u + 2n\omega) \times \sin \mathrm{A}(2n\omega - u)$, will be found by the formula (A) of § 2, observing that $\sin a = \sin \mathrm{A}(2n\omega) = \sin \lambda_{2n}$, $\sin b = \sin \mathrm{A} u = \sin \phi$, $\sin s = \sin \mathrm{A}(2n\omega + u)$, $\sin \sigma = \sin(2n\omega - u)$:
thus we have,

$$\sin \mathrm{A}(u + 2n\omega) \sin \mathrm{A}(u + 2p\omega - 2n\omega) = \frac{\sin^2 \lambda_{2n} - \sin^2 \phi}{1 - k^2 \sin^2 \lambda_{2n} \sin^2 \phi}.$$

Wherefore, by taking in all the factors and writing z for $\sin\phi$, we shall obtain,

$$\sin \psi = \frac{z}{\gamma} \cdot \frac{\sin^2 \lambda_2 - z^2}{1 - k^2 z^2 \sin^2 \lambda_2} \cdot \frac{\sin^2 \lambda_4 - z^2}{1 - k^2 z^2 \sin^2 \lambda_4} \cdots \frac{\sin^2 \lambda_{p-1} - z^2}{1 - k^2 z^2 \sin^2 \lambda_{p-1}}.$$

The expression of γ, viz.

$$\gamma = \sin \lambda_1 . \sin \lambda_3 . \sin \lambda_5 \ldots \sin \lambda_{2p-1},$$

may be written in this form,

$$\gamma = \sin^2 \lambda_1 . \sin^2 \lambda_3 . \sin^2 \lambda_5 \ldots . \sin^2 \lambda_{p-1},$$

omitting the factor $\sin \lambda_p = 1$: wherefore, if we assume,

$$\beta = \frac{\sin^2 \lambda_2 \cdot \sin^2 \lambda_4 \cdot \sin^2 \lambda_6 \ldots \sin^2 \lambda_{p-1}}{\sin^2 \lambda_1 \cdot \sin^2 \lambda_3 \cdot \sin^2 \lambda_5 \ldots \sin \lambda_{p-2}},$$

we shall have,

p being an odd number,

$$\sin \psi = \beta z \cdot \frac{1 - \frac{z^2}{\sin^2 \lambda_2}}{1 - k^2 z^2 \sin^2 \lambda_2} \cdot \frac{1 - \frac{z^2}{\sin^2 \lambda_4}}{1 - k^2 z^2 \sin^2 \lambda_4} \cdots \frac{1 - \frac{z^2}{\sin^2 \lambda_{p-1}}}{1 - k^2 z^2 \sin^2 \lambda_{p-1}}. \tag{2}$$

When p is an even number, if we leave out the first factor in the numerator of the expression of y or $\sin \psi$, there will remain an odd number of factors, that which occupies the middle place, being $\sin \text{A} (u + p \omega)$: and any factor, as $\sin \text{A} (u + 2 n \omega)$, between the first and the middle one, will have another, viz. $\sin \text{A} (u + 2 p \omega - 2 n \omega)$, corresponding to it after the middle one; and the product of this pair of factors will be obtained as before, viz.

$$\sin \text{A} (u + 2 n \omega) \sin \text{A} (u + 2 p \omega - 2 n \omega) = \frac{\sin^2 \lambda_{2n} - \sin^2 \phi}{1 - k^2 \sin^2 \lambda_{2n} \sin^2 \phi}.$$

With regard to the middle factor, we shall have, in the formulas of § 2, $\sin a = \sin \text{A} (p \omega) = \sin 90°$, $\sin b = \sin \text{A} u = \sin \phi$, $\sin s = \sin \text{A} (u + p \omega)$; and

$$\sin \text{A} (u + p \omega) = \frac{\cos \phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

Wherefore, by proceeding as before, we shall have,

p being an even number,

$$\sin \psi = \frac{\beta z \sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}} \cdot \frac{1 - \frac{z^2}{\sin^2 \lambda_2}}{1 - k^2 z^2 \sin^2 \lambda_2} \cdot \frac{1 - \frac{z^2}{\sin^2 \lambda_4}}{1 - k^2 z^2 \sin^2 \lambda_4} \cdots \frac{1 - \frac{z^2}{\sin^2 \lambda_{p-2}}}{1 - k^2 z^2 \sin^2 \lambda_{p-2}} \tag{3}$$

$$\beta = \frac{\sin^2 \lambda_2 \cdot \sin^2 \lambda_4 \cdot \sin^2 \lambda_6 \ldots \sin^2 \lambda_{p-2}}{\sin^2 \lambda_1 \cdot \sin^2 \lambda_3 \cdot \sin^2 \lambda_5 \ldots \sin^2 \lambda_{p-1}}.$$

In both the formulas (2) and (3), it is obvious that β is the quotient of the product of the sines of all the even amplitudes, λ_2, λ_4, &c. between the limits 0 and 180°, divided by the product of the sines of all the odd amplitudes, λ_1, λ_3, &c. contained between the same limits. The general expression of β, common to the two cases, is therefore as follows,

$$\beta = \frac{\sin \lambda_2 \cdot \sin \lambda_4 \cdot \sin \lambda_6 \ldots \sin \lambda_{2p-2}}{\sin \lambda_1 \cdot \sin \lambda_3 \cdot \sin \lambda_5 \ldots \sin \lambda_{2p-1}}. \tag{4}$$

In the formula (2) let P and R stand for the products of the binomials in the numerator and denominator; then,

$$\sin\psi = \frac{\beta z P}{R};$$

and,

$$\cos^2\psi = \frac{R^2 - \beta^2 z^2 P^2}{R^2}.$$

The numerator of this expression is a rational function of z^2, and it will vanish whenever $\cos^2\psi = 0$, or $\sin^2\psi = 1$, that is, when z^2 is equal to $\sin^2\lambda_{2n+1}$, $2n+1$ being any odd number less than $2p$. Suppose that $2n+1$ is any odd number less than p, the numerator of the value of $\cos^2\psi$ will be divisible by $\left(1 - \frac{z^2}{\sin^2\lambda_{2n+1}}\right)$, and also by $\left(1 - \frac{z^2}{\sin^2\lambda_{2p-2n-1}}\right)$; and as these binomials are equal, it will be divisible by their product $\left(1 - \frac{z^2}{\sin^2\lambda_{2n+1}}\right)^2$; and, p being itself an odd number, to the double divisors there must be added the single one $\left(1 - \frac{z^2}{\sin^2\lambda_p}\right) = 1 - z^2$. The numerator is therefore divisible by the product,

$$(1 - z^2) \cdot \left(1 - \frac{z^2}{\sin^2\lambda_1}\right)^2 \cdot \left(1 - \frac{z^2}{\sin^2\lambda_3}\right)^2 \cdots \left(1 - \frac{z^2}{\sin^2\lambda_{p-2}}\right)^2 :$$

and, as the two expressions have the same absolute term and the same dimensions, they must be identical. Wherefore we have,

p being an odd number,

$$\cos\psi = \sqrt{1 - z^2} \cdot \frac{1 - \frac{z^2}{\sin^2\lambda_1}}{1 - k^2 z^2 \sin^2\lambda_2} \cdot \frac{1 - \frac{z^2}{\sin^2\lambda_3}}{1 - k^2 z^2 \sin^2\lambda_4} \cdots \frac{1 - \frac{z^2}{\sin^2\lambda_{p-2}}}{1 - k^2 z^2 \sin^2\lambda_{p-1}}. \qquad (5)$$

In like manner, if P and R represent the rational binomial products in the numerator and denominator of the formula (3), we shall have

$$\sin\psi = \frac{\beta z \sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}} \times \frac{P}{R};$$

and

$$\cos^2\psi = \frac{(1 - k^2 z^2) R^2 - \beta^2 z^2 (1 - z^2) P^2}{(1 - k^2 z^2) R^2}.$$

Proceeding as before, it will appear that the numerator of this expression is divisible by the double divisor $\left(1-\frac{z^2}{\sin^2\lambda_{2n+1}}\right)^2$, $2n+1$ being any odd number less than p; and in this case when p is an even number, all the divisors are double. Wherefore the product

$$\left(1-\frac{z^2}{\sin^2\lambda_1}\right)^2 \cdot \left(1-\frac{z^2}{\sin^2\lambda_3}\right)^2 \cdot \left(1-\frac{z^2}{\sin^2\lambda_5}\right)^2 \cdots \left(1-\frac{z^2}{\sin^2\lambda_{p-1}}\right)^2$$

will divide the numerator of the value of $\cos^2\psi$; and it will be identical to it, because both the expressions have the same dimensions. Thus we obtain,

p being an even number,

$$\cos\psi = \frac{1-\frac{z^2}{\sin^2\lambda_1}}{\sqrt{1-k^2z^2}} \cdot \frac{1-\frac{z^2}{\sin^2\lambda_3}}{1-k^2z^2\sin^2\lambda_2} \cdots\cdots \frac{1-\frac{z^2}{\sin^2\lambda_{p-1}}}{1-k^2z^2\sin^2\lambda_{p-2}}. \qquad (6)$$

From the equations (2) and (5) we deduce,

$$\tan\psi = \frac{\beta z}{\sqrt{1-z^2}} \cdot \frac{1-\frac{z^2}{\sin^2\lambda_2}}{1-\frac{z^2}{\sin^2\lambda_1}} \cdot \frac{1-\frac{z^2}{\sin^2\lambda_4}}{1-\frac{z^2}{\sin^2\lambda_3}} \cdots \frac{1-\frac{z^2}{\sin^2\lambda_{p-1}}}{1-\frac{z^2}{\sin^2\lambda_{p-2}}}$$

but it will readily appear that

$$\frac{1-\frac{\sin^2\phi}{\sin^2\lambda_{2n}}}{1-\frac{\sin^2\phi}{\sin^2\lambda_{2n+1}}} = \frac{1-\frac{\tan^2\phi}{\tan^2\lambda_{2n}}}{1-\frac{\tan^2\phi}{\tan^2\lambda_{2n+1}}}:$$

wherefore we obtain,

p being an odd number,

$$\tan\psi = \beta\tan\phi \times \frac{1-\frac{\tan^2\phi}{\tan^2\lambda_2}}{1-\frac{\tan^2\phi}{\tan^2\lambda_1}} \cdot \frac{1-\frac{\tan^2\phi}{\tan^2\lambda_4}}{1-\frac{\tan^2\phi}{\tan^2\lambda_3}} \cdots \frac{1-\frac{\tan^2\phi}{\tan^2\lambda_{p-1}}}{1-\frac{\tan^2\phi}{\tan^2\lambda_{p-2}}}. \qquad (7)$$

And in a similar manner we deduce from the equations (3) and (6),

p being an even number,

$$\tan\psi = \frac{\beta\tan\phi}{1-\frac{\tan^2\phi}{\tan^2\lambda_1}} \cdot \frac{1-\frac{\tan^2\phi}{\tan^2\lambda_2}}{1-\frac{\tan^2\phi}{\tan^2\lambda_3}} \cdot \frac{1-\frac{\tan^2\phi}{\tan^2\lambda_4}}{1-\frac{\tan^2\phi}{\tan^2\lambda_5}} \cdots \frac{1-\frac{\tan^2\phi}{\tan^2\lambda_{p-2}}}{1-\frac{\tan^2\phi}{\tan^2\lambda_{p-1}}}. \qquad (8)$$

The formulas (2), (5), (7), in which p is an odd number, are those used in the theorems of M. Jacobi; the other three, (3), (6), and (8), have been added here. All the formulas will be true in the circle, if we make $k = 0$, and derive the arcs λ_1, λ_2, &c. from the subdivision of the quadrant, in like manner as they have been obtained from the subdivision of the definite integral $k\left(\frac{\pi}{2}\right)$. The coefficient β is the expression of the first ratio of the nascent arcs ψ and φ; and it is equal to p in the circle.

All the formulas are, however, imperfect in one respect: they all suppose that the amplitudes λ_1, λ_2, &c., derived from the subdivision of the definite integral $k\left(\frac{\pi}{2}\right)$, are known. By means of these amplitudes, the general solution of the problem has been deduced from a particular case: but the formulas cannot be considered as complete till all the coefficients have been expressed in functions of the modulus k; and, with respect to this point, the researches of analysts have not yet been entirely successful.

5. Having now investigated the relation between the arcs ψ and φ, we have next to demonstrate that the equation (1) is true when these amplitudes are substituted in it, and a proper value is assigned to the indeterminate modulus h; but this requires some preparation, in order to avoid complicated operations.

First, p being an odd number, we have,

$$\sin\psi = \frac{\beta \cdot z\mathrm{P}}{\mathrm{R}}, \quad \cos\psi = \frac{\sqrt{1-z^2} \cdot \mathrm{Q}}{\mathrm{R}};$$

R, P, Q, representing the rational binomial products in the denominators and numerators of the equations (2) and (5): we therefore obtain,

$$\mathrm{R}^2 = \beta^2 z^2 \mathrm{P}^2 + (1 - z^2)\,\mathrm{Q}^2.$$

This equation has been found on the supposition that z is less than 1; but, as it contains no radical quantities, it will be true for all values of z. We may therefore substitute $\frac{1}{kz}$ for z; and, in the resulting equation, the symbol z will still represent a quantity unrestricted in its value. Now, the substitution of $\frac{1}{kz}$ for z being made, we shall obtain,

$$\mathrm{R}^2 - \beta^2 h^2 z^2 \mathrm{P}^2 = (1 - k^2 z^2)\,\mathrm{R}'^2,$$

in which expression R and P denote the same functions of z as before, and the values of the new symbols h and R′ are as follows,

$$h = k^p \sin^4 \lambda_1 \,.\, \sin^4 \lambda_3 \,.\, \sin^4 \lambda_5 \ldots \sin^4 \lambda_{p-2},$$

$$R' = (1 - k^2 z^2 \sin^2 \lambda_1)(1 - k^2 z^2 \sin^2 \lambda_3) \ldots (1 - k^2 z^2 \sin^2 \lambda_{p-2}).$$

We thus have

$$\left.\begin{array}{l} \sqrt{R^2 - \beta^2 z^2 P^2} = \sqrt{1 - z^2} \,.\, Q \\ \sqrt{R^2 - \beta^2 h^2 z^2 P^2} = \sqrt{1 - k^2 z^2} \,.\, R'. \end{array}\right\} \qquad \text{(C)}$$

Secondly, when p is an even number, R, P, Q will stand for the rational binomial products in the denominators and numerators of the equations (3) and (6): thus

$$\sin \psi = \frac{\beta z \sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}} \cdot \frac{P}{R}, \quad \cos \psi = \frac{1}{\sqrt{1 - k^2 z^2}} \cdot \frac{Q}{R}:$$

consequently

$$(1 - k^2 z^2) R^2 = \beta^2 z^2 (1 - z^2) P^2 + Q^2.$$

And if in this equation we substitute $\frac{1}{kz}$ in place of z, we shall obtain this result,

$$(1 - k^2 z^2) R^2 = \beta^2 z^2 (1 - z^2) P^2 + R'^2$$

R and P representing the same functions of z as before, and the new symbols h and R′ standing for these values,

$$h = k^p \,.\, \sin^4 \lambda_1 \,.\, \sin^4 \lambda_3 \,.\, \sin^4 \lambda_5 \ldots \sin^4 \lambda_{p-1}$$

$$R' = (1 - k^2 z^2 \sin^2 \lambda_1)(1 - k^2 z^2 \sin^2 \lambda_3) \ldots (1 - k^2 z^2 \sin^2 \lambda_{p-1}.$$

From what has been proved we now have

$$\left.\begin{array}{l} \sqrt{(1 - k^2 z^2) R^2 - \beta^2 z^2 (1 - z^2) P^2} = Q, \\ \sqrt{(1 - k^2 z^2) R^2 - \beta^2 h^2 z^2 (1 - z^2) P^2} = R' \end{array}\right\} \qquad \text{(D)}$$

To these formulas we must add the following principle of analysis, on which the demonstration we have in view mainly turns. Let V and U denote rational functions of z: we shall have this identical equation,

$$\frac{d.\left(\frac{V}{U}\right)}{dz} U^2 = \frac{1}{a}\left\{ (V + aU)\frac{d.(V - aU)}{dz} - (V - aU)\frac{d.(V + aU)}{dz}\right\}:$$

from which it follows, that every double binomial factor either of $V + aU$, or of $V - aU$, is a simple binomial factor of $\frac{d.\left(\frac{V}{U}\right)}{dz} U^2$; and further, if $V + aU$ and $V - aU$ have no common divisor, that every double binomial factor of $(V + aU) \times (V - aU) = V^2 - a^2 U^2$, is a simple binomial factor of $\frac{d.\left(\frac{V}{U}\right)}{dz} . U^2$.

6. The differential of the equation (1) may now be readily demonstrated, supposing that β has the value investigated in § 4, and h, the value assigned to it in § 5. And first when p is an odd number, we obtain from the equation (2),

$$\sin \psi = \frac{\beta . z P}{R}, \quad z = \sin \phi :$$

and with these values the equation (1) will become

$$\frac{\frac{R^2}{dz} d.\left(\frac{zP}{R}\right)}{\sqrt{(R^2 - \beta^2 z^2 P^2)(R^2 - \beta^2 h^2 z^2 P^2}} = \frac{1}{\sqrt{1 - z^2 . 1 - k^2 z^2}}:$$

and, on account of the formulas (C),

$$\frac{\frac{R^2}{dz} d.\left(\frac{zP}{R}\right)}{Q . R'} = 1.$$

Now it is evident that $R + \beta . zP$ and $R - \beta . zP$, have no common divisor: for, as R contains only the even powers of z, and zP only the odd powers, if $1 + cz$ be a factor of $R + \beta . zP$, $1 - cz$ will necessarily be a factor of $R - \beta . zP$. Wherefore, according to what has been proved above, every double binomial factor of $R^2 - \beta^2 z^2 P^2$, that is, every factor of Q, will be a factor of the function in the numerator of the left side of the last equation. In the very same manner it is proved that every double binomial factor of $R^2 - \beta^2 h^2 z^2 P^2$, that is, every factor of R', will be a factor of the same function.

Wherefore the numerator of the left side of the last equation is divisible by the product $Q \times R'$ in the denominator; and, as both the expressions have the same dimensions and the same absolute term, they are identical; which verifies the equation. Wherefore the equation (1) is demonstrated when p is an odd number.

Secondly, when p is an even number, we have by equation (3),

$$\sin \psi = \frac{\beta z \sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}} \cdot \frac{P}{R}, \ \sin \phi = z:$$

and the differential of equation (1) will become by substitution,

$$\frac{\frac{(1 - k^2 z^2) R^2}{dz} \cdot d \cdot \left(\frac{z \sqrt{1 - z^2} . P}{\sqrt{1 - k^2 z^2} . R}\right)}{\sqrt{\left((1 - k^2 z^2) R^2 - \beta^2 z^2 (1 - z^2) P^2\right)\left((1 - k^2 z^2) R^2 - \beta^2 h^2 z^2 (1 - z^2) P^2\right)}}$$

$$= \frac{1}{\sqrt{1 - z^2 . 1 - k^2 z^2}} \text{ : and, on account of the formulas (D),}$$

$$\frac{\frac{(1 - k^2 z^2) R^2}{dz} d \cdot \left(\frac{z \sqrt{1 - z^2} . P}{\sqrt{1 - k^2 z^2} . R}\right)}{Q . R'} = \frac{1}{\sqrt{1 - z^2 . 1 - k^2 z^2}}.$$

It will be proved, by the like reasoning as before, that the numerator of the left side of this equation is divisible by the product in the denominator. Now if we perform the differentiation indicated, we shall find,

$$S = (1 - 2 z^2 + k^2 z^4) P R + z (1 - z^2)(1 - k^2 z^2) R^2 \frac{d\left(\frac{P}{R}\right)}{dz},$$

$$\frac{(1 - k^2 z^2) R^2}{dz} d \cdot \left(\frac{z \sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}} \cdot \frac{P}{R}\right) = \frac{S}{\sqrt{1 - z^2 . 1 - k^2 z^2}}:$$

and it is evident that all the rational factors of the left side of this last formula, and consequently all the factors of $Q \times R'$, will be factors of S. By substitution the differential equation (1) will now become

$$\frac{S}{Q \times R'} = 1,$$

which is manifestly verified: for, as $Q \times R'$ divides S, and the two expressions have the same dimensions and the same absolute term, they are identical. The equation (1) is therefore demonstrated when p is an even number.

7. The transformation expressed by the equation,

$$\int_0^{\psi} \frac{d\psi}{\sqrt{1 - h^2 \sin^2 \psi}} = \beta \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

has now been demonstrated for any number whether odd or even, the constant β being determined by the formula (4), and the modulus h by the special formulas in § 5, or, generally without distinguishing whether p is odd or even, by this formula,

$$h = k^p \,.\, (\sin \lambda_1 \sin \lambda_3 \sin \lambda_5 \ldots \sin \lambda_{2p-1})^2, \qquad (9)$$

the sines multiplied together being those of all the odd amplitudes less than 180°. The relation between the variable amplitudes ψ and φ is expressed by the several equations in § 4.

In order to render the solution of the problem more complete, it may be proper to add a useful method of computing the amplitude ψ.

In § 5 we have obtained this equation,

$$R^2 = \beta^2 z^2 P^2 + (1 - z^2) Q^2.$$

And, if we represent by N and M the products of the binomials in the numerator and denominator of the equation (7), we shall have

$$\tan \psi = \frac{\beta \tan \varphi \, N}{M}.$$

Let $x = \tan \varphi$, then $z^2 = \sin^2 \varphi = \frac{x^2}{1 + x^2}$; and, observing that

$$1 - \frac{z^2}{\sin^2 \lambda_n} = \frac{1 - \frac{x^2}{\tan^2 \lambda_n}}{(1 + x^2)},$$

it will readily appear that

$$\beta z P = \frac{\beta x N}{(1 + x^2)^{\frac{p}{2}}}, \quad \sqrt{1 - z^2} \,.\, Q = \frac{M}{(1 + x^2)^{\frac{p}{2}}}.$$

These values being substituted in the foregoing equation, we get

$$M^2 + \beta^2 x^2 N^2 = (1 + x^2)^p \,.\, R^2:$$

And if R be transformed into a function of x^2, we shall obtain

$$M^2 + \beta^2 x^2 N^2 = (1 + x^2)(1 + c_2^2 x^2)^2 (1 + c_4^2 x^2)^2 \ldots (1 + c^2_{p-2} x^2)^2,$$

the new symbol c_{2n} being determined by this formula,

$$\overline{c_{2n}}|^2 = 1 - k^2 \sin^2 \lambda_{2n}.$$

The last equation may be resolved into these two,

$$M + \beta x N \sqrt{-1} = (1 + x\sqrt{-1})(1 + c_2 x \sqrt{-1})^2 \ldots (1 + c_{p-2} x \sqrt{-1})^2,$$
$$M - \beta x N \sqrt{-1} = (1 - x\sqrt{-1})(1 - c_2 x \sqrt{-1})^2 \ldots (1 - c_{p-2} x \sqrt{-1})^2:$$

the second of which being divided by the first, there will result,

$$\frac{1 - \tan\psi\sqrt{-1}}{1 + \tan\psi\sqrt{-1}} = \frac{1 - x\sqrt{-1}}{1 + x\sqrt{-1}} \cdot \left(\frac{1 - c_2 x\sqrt{-1}}{1 + c_2 x \sqrt{-1}}\right)^2 \ldots \left(\frac{1 - c_{p-2} x \sqrt{-1}}{1 + c_{p-2} x \sqrt{-1}}\right)^2.$$

Now u being an arc of a circle, we have this well known formula,

$$u = \frac{1}{2\sqrt{-1}} \times \log. \left(\frac{1 - \tan u \sqrt{-1}}{1 + \tan u \sqrt{-1}}\right):$$

wherefore, if we take the logarithms of the factors of the foregoing expression, and substitute the equivalent circular arcs, we shall obtain,

p being an odd number,

$$\psi = \phi + 2\phi_2 + 2\phi_4 + 2\phi_6 \ldots 2\phi_{p-1}, \qquad (10)$$

the arc ϕ_{2n} being determined by the equation,

$$\tan \phi_{2n} = c_{2n} \times \tan \phi.$$

When p is an even number, we have this equation in § 5,

$$(1 - k^2 z^2) R^2 = \beta^2 z^2 (1 - z^2) P^2 + Q^2.$$

And, using N and M to denote the products of the binomials in the numerator and denominator of the equation (8), we have

$$\tan \psi = \frac{\beta \tan \phi N}{M}.$$

By the substitution of $\frac{x}{\sqrt{1 + x^2}}$ for z as before, it will be found that,

$$\beta z \sqrt{1 - z^2} . P = \frac{\beta x N}{(1 + x^2)^{\frac{p}{2}}}, \quad Q = \frac{M}{(1 + x^2)^{\frac{p}{2}}}.$$

Wherefore we have,

$$M^2 + \beta^2 x^2 N^2 = (1 + x^2)^p (1 - k^2 z^2) R^2;$$

and by converting $(1 - k^2 z^2) . R^2$ into a function of x^2, we get

$$M^2 + \beta^2 x^2 N^2 = (1 + x^2)(1 + k'^2 x^2)(1 + c_2^2 x^2)^2 \dots (1 + c^2_{p-2} x^2)^2,$$

$$k'^2 = 1 - k^2, \quad \overline{c_{2n}}|^2 = 1 - k^2 \sin^2 \lambda_{2n}.$$

By treating this equation as before, we get

$$\frac{1 - \tan\psi\sqrt{-1}}{1 + \tan\psi\sqrt{-1}} = \frac{1 - x\sqrt{-1}}{1 + x\sqrt{-1}} \cdot \frac{1 - k' x\sqrt{-1}}{1 + k' x\sqrt{-1}} \left(\frac{1 - c_2 x\sqrt{-1}}{1 + c_2 x\sqrt{-1}}\right)^2 \dots :$$

and from this we deduce,

p being an even number,

$$\psi = \varphi + \varphi' + \varphi_2 + \varphi_4 \dots + \varphi_{p-2}, \quad (11)$$

$$\tan\varphi' = k' \tan\varphi, \quad \tan\varphi_{2n} = c_{2n} \tan\varphi.$$

8. In what goes before, our attention has been confined to two related functions, which, for the sake of abridging, we have denoted by the prefixes H and K; but as we shall have occasion, in what follows, to compare several functions differing from one another in their moduli and amplitudes, it will be proper to adopt the usual and more general notation, by means of the characteristic F prefixed to the modulus and amplitude. According to this notation, the equation (1) will be thus written,

$$F(h, \psi) = \beta F(k, \varphi); \quad F(k, \varphi) = \frac{1}{\beta} F(h, \psi).$$

The modulus k being given, we can compute the amplitudes, λ_1, λ_2, &c., at least by approximation; and the amplitude φ being supposed known, the foregoing formulas will determine the modulus h, the multiplier β, and the amplitude ψ; so that the function $F(k, \varphi)$ will be reduced to the similar function $F(h, \psi)$, of which the modulus h is less than the given modulus k. And in like manner as the three quantities h, β, ψ were determined from the two k, φ, we can deduce, from the two h, ψ, three new quantities, $h_{\prime}$, $\beta_{\prime}$, $\psi_{\prime}$, which will satisfy the equations,

$$F(h_{\prime}, \psi_{\prime}) = \beta_{\prime} F(h, \psi); \quad F(k, \varphi) = \frac{1}{\beta\beta_{\prime}} \times F(h_{\prime}, \psi_{\prime});$$

the modulus $h_{\prime}$ being less than the modulus h. Continuing the like operations, we can pass along a scale of decreasing moduli, till we arrive at one which, being as small as we please, will make the function $F(k, \phi)$ approach to a circular arc as near as may be required.

If we wish to apply the same theorem to reduce the given function $F(h, \phi)$ to a logarithm, through a scale of increasing moduli, the process is not so direct. For, in the first place, the greater modulus k is not immediately deducible from the less h, by means of the formulas that have been investigated; and, in the second place, the amplitude ϕ cannot be found when ψ is given without solving an equation of p dimensions. The theorem is, no doubt, mathematically sufficient for effecting the reduction; but the operations required are practically impossible, except in a few cases when p is a small number. But the ingenuity of M. Jacobi has provided a remedy for this inconvenience by a new transformation, which we shall now briefly explain, as it discloses a new set of remarkable properties of the elliptic functions.

If we put $y = \tan \psi$, $x = \tan \phi$, $h'^2 = 1 - h^2$, $k'^2 = 1 - k^2$, the differential of the equation (1) will assume this form,

$$\frac{dy}{\sqrt{1 + y^2 \cdot 1 + h'^2 y^2}} = \frac{\beta\, dx}{\sqrt{1 + x^2 \cdot 1 + k'^2 x^2}}:$$

and, for solving this equation, we shall have by the formula (7),

p being an odd number,

$$y = \beta x \times \frac{1 - \dfrac{x^2}{\tan^2 \lambda_2}}{1 - \dfrac{x^2}{\tan^2 \lambda_1}} \cdot \frac{1 - \dfrac{x^2}{\tan^2 \lambda_4}}{1 - \dfrac{x^2}{\tan^2 \lambda_3}} \cdots \frac{1 - \dfrac{x^2}{\tan^2 \lambda_{p-1}}}{1 - \dfrac{x^2}{\tan^2 \lambda_{p-2}}}.$$

But if this value of y solve the differential equation, it will still solve it, if we change $+x^2$ and $+y^2$ into $-x^2$ and $-y^2$; for it is obvious that, if the expression of y make the two sides of the equation identical in one case, it will necessarily make them identical in the other case. Wherefore the equation

$$\frac{dy}{\sqrt{1 - y^2 \cdot 1 - h'^2 y^2}} = \frac{\beta\, dx}{\sqrt{1 - x^2 \cdot 1 - k'^2 x^2}},$$

will have for its solution.

$$y = \beta x \times \frac{1 + \frac{x^2}{\tan^2 \lambda_2}}{1 + \frac{x^2}{\tan^2 \lambda_1}} \cdot \frac{1 + \frac{x^2}{\tan^2 \lambda_4}}{1 + \frac{x^2}{\tan^2 \lambda_3}} \cdots \frac{1 + \frac{x^2}{\tan^2 \lambda_{p-1}}}{1 + \frac{x^2}{\tan^2 \lambda_{p-1}}}.$$

In this equation the values of y and x are between 0 and ± 1, which limits they both attain at the same time. If we make $x = \pm 1$, and attend to the value of β, we shall find $y = \pm 1$. Let $y = \sin \tau$, $x = \sin \sigma$: then the integral of the differential equation will be

$$\left.\begin{array}{l} F(h', \tau) = \beta F(k', \sigma) \\ \sin \tau = \beta \sin \sigma \times \dfrac{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_2}}{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_1}} \cdot \dfrac{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_4}}{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_3}} \cdots \dfrac{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_{p-1}}}{1 + \frac{\sin^2 \sigma}{\tan^2 \lambda_{p-2}}}: \end{array}\right\} \quad (12)$$

the amplitudes τ and σ increasing together from zero, and becoming equal to one another at 90°, and at every multiple of 90°.

A property of considerable importance in this theory, results from the comparison of the equations (1) and (12). Recalling the notations before used, viz. $K = F\left(k, \frac{\pi}{2}\right)$ and $H = F\left(h, \frac{\pi}{2}\right)$, we obtain from what has already been said in § 1,

$$p \times H = \beta \times K:$$

and if we put similarly $K' = F\left(k', \frac{\pi}{2}\right)$ and $H' = F\left(h', \frac{\pi}{2}\right)$, and observe that in the equations (12), τ and σ are equal to 90° at the same time, we shall have,

$$H' = \beta K'.$$

By combining the two equations, we readily obtain, first,

$$\frac{H}{H'} = \frac{1}{p} \cdot \frac{K}{K'}; \ \beta = p \cdot \frac{H}{K}; \quad (13)$$

and secondly,

$$\beta \beta' = p; \quad \frac{K'}{K} = \frac{1}{p} \cdot \frac{H'}{H}; \quad \beta' = p \cdot \frac{K'}{H'}. \quad (14)$$

For any number p, the first of the formulas (13) determines h, and the second determines β, when k is given. Both the formulas involve transcendent quantities; they are nevertheless of great practical utility in this theory; and they express succinctly the conditions necessary, in order that the transformations in the equations (1) and (12) take place. A little attention will show that the formulas (14) and (13) are entirely similar, the quantities β', h', k' occupying the same places in the first, that β, k, h do in the other. From this we learn that the equations (1) and (12) will still be true if we change β, k, h for β', h', k', respectively. Thus we have,

$$F(k', \psi) = \beta' F(h', \phi), \tag{15}$$

the letters ψ and ϕ, it need hardly be noticed, although used on a former occasion, here express simply the variable amplitudes of the related functions. If therefore we divide $H' = F\left(h', \frac{\pi}{2}\right)$ into p equal parts, and put μ_1, μ_2, μ_3, &c., for the respective amplitudes of $\frac{1}{p}H'$, $\frac{2}{p}H'$, $\frac{3}{p}H'$, &c.; we shall have by the formulas (4) and (9),

$$\left.\begin{aligned} \beta' &= \frac{\sin\mu_2 \sin\mu_4 \ldots \sin\mu_{2p-2}}{\sin\mu_1 \sin\mu_3 \ldots \sin\mu_{2p-1}}, \\ k' &= h'^{p}.\left(\sin\mu_1 \sin\mu_3 \ldots \sin\mu_{2p-1}\right)^2. \end{aligned}\right\} \tag{16}$$

The multipliers β and β' being similar functions, the first of the amplitudes λ_1, λ_2, λ_3, &c., and the other of the amplitudes μ_1, μ_2, μ_3, &c., the equation $\beta\beta' = p$, expresses a curious property of those functions.

And, in like manner, if we change β, k, h, respectively for β', h', k' in the equation (12); or, which is the same thing, if we derive an equation from (15) in the same manner that (12) was obtained from (1), we shall get

$$\left.\begin{aligned} &F(k, \tau) = \beta' F(h, \sigma), \\ &\sin\tau = \beta' \sin\sigma \times \frac{1 + \frac{\sin^2\sigma}{\tan^2\mu_2}}{1 + \frac{\sin^2\sigma}{\tan^2\mu_1}} \cdot \frac{1 + \frac{\sin^2\sigma}{\tan^2\mu_4}}{1 + \frac{\sin^2\sigma}{\tan^2\mu_3}} \cdots \frac{1 + \frac{\sin^2\sigma}{\tan^2\mu_{p-1}}}{1 + \frac{\sin^2\sigma}{\tan^2\mu_{p-2}}}. \end{aligned}\right\} \tag{17}$$

Although, in the investigations of this §, we have supposed that p is an odd number, yet it is obvious that they will succeed equally when p is an even number, the formula (8) being used instead of (7).

The analysis by which the equation (12), of which those that follow are consequences, has been deduced from the equation (1), is precisely that by which the expression of a circular arc is made to pass into a logarithm; so that the whole of this analytical theory rests on one principle, namely, the analogy which an elliptic function bears to a circular arc and to a logarithm, which are its extreme limits.

10. Of the transformations in the last §, the principal one is contained in the formulas (17), which constitute what is called the second theorem of M. JACOBI. One of its chief uses is to supply the defect of the first theorem by furnishing a direct process for reducing an elliptic function to a logarithm, through a scale of increasing moduli. In the function $F(h, \sigma)$, the modulus h being given, we know h' $(= \sqrt{1 - h^2})$ named the complement of h for the sake of abridging; we shall therefore obtain the amplitudes μ_1, μ_2, &c., by the subdivision of the function $H' = F\left(h', \frac{\pi}{2}\right)$; we next compute the quantities β' and k' by the formulas (16); and, the amplitude σ being given, we deduce from the formulas (17), the amplitude τ, which will satisfy the equation,

$$F(h, \sigma) = \frac{1}{\beta'} \cdot F(k, \tau),$$

the modulus k of the new function being greater than h, because the comple ment k' is less than the complement h'. Taking now k' the complement of k, we deduce from it, by means of the formulas (16) and (17), the three quantities $\beta'_{\prime}$, $k'_{\prime}$, $\tau_{\prime}$, in like manner as β', k', τ were deduced from h'; and we shall have these equations,

$$F(k, \tau) = \beta'_{\prime} F(k_{\prime}, \tau_{\prime})\,;\ F(h, \sigma) = \frac{1}{\beta' \beta'_{\prime}} \times F(k_{\prime}, \tau_{\prime});$$

the modulus $k_{\prime}$ being greater than k, because the complement $k'_{\prime}$ is less than k'. The like operations being continued, we shall at length arrive at a modulus k_n, as near the limit 1 as may be required.

Another use of the second theorem, when combined with the first, is to find any multiple of an elliptic function, or any aliquot part of it. By the first theorem, we have

$$F(h, \psi) = \beta\, F(k, \phi)\,;$$

and by the second, making $\sigma = \psi$ in the equations (17),

$$F(k, \tau) = \beta'\, F(h, \psi)\,;$$

and, by combining the two equations, observing that $\beta\beta' = p$, we get

$$F(k, \tau) = p\,F(k, \phi).$$

If p be an odd number, the amplitudes are obtained by the formulas (2) and (17), viz.

$$\sin\psi = \beta\sin\phi \times \frac{1-\dfrac{\sin^2\phi}{\sin^2\lambda_2}}{1-k^2\sin^2\phi\sin^2\lambda_2}\cdots\cdots\frac{1-\dfrac{\sin^2\phi}{\sin^2\lambda_{p-1}}}{1-k^2\sin^2\phi\sin^2\lambda_{p-1}},$$

$$\sin\tau = \beta'\sin\psi \times \frac{1+\dfrac{\sin^2\psi}{\tan^2\mu_2}}{1+\dfrac{\sin^2\psi}{\tan^2\mu_1}}\cdots\cdots\cdots\cdots\frac{1+\dfrac{\sin^2\psi}{\tan^2\mu_{p-1}}}{1+\dfrac{\sin^2\psi}{\tan^2\mu_{p-2}}}.$$

When a multiple is required, we pass directly, by means of the two equations, from the given amplitude ϕ to τ which is sought. In the case of an aliquot part, the amplitude τ being given, the solution of the second equation, of which p is the dimensions, will determine $\sin\psi$; and the amplitude ϕ which is sought, will then be found by solving the first equation, which is also of p dimensions. From the nature of the second equation, it has only one real root, and $p-1$ impossible roots, for every real value of $\sin\tau$; and therefore it follows from the first equation, that the amplitude ϕ of the function $\frac{1}{p}F(k, \tau)$ admits in all of p^2 values, of which only p values are real quantities, and the rest impossible.

If p be an even number, the expression of $\sin\psi$ will contain radical quantities, but instead of it we may take the value of $\tan\psi$ in the formula (8); and the two equations for the amplitudes will be,

$$\tan\psi = \frac{\beta\tan\phi}{1-\dfrac{\tan^2\phi}{\tan^2\lambda_1}}\cdot\frac{1-\dfrac{\tan^2\phi}{\tan^2\lambda_2}}{1-\dfrac{\tan^2\phi}{\tan^2\lambda_3}}\cdots\cdots\frac{1-\dfrac{\tan^2\phi}{\tan^2\lambda_{p-2}}}{1-\dfrac{\tan^2\phi}{\tan^2\lambda_{p-1}}},$$

$$\sin\tau = \frac{\beta'\sin\psi}{1+\dfrac{\sin^2\psi}{\tan^2\mu_1}}\cdot\frac{1+\dfrac{\sin^2\psi}{\tan^2\mu_2}}{1+\dfrac{\sin^2\psi}{\tan^2\mu_3}}\cdots\cdots\frac{1+\dfrac{\sin^2\psi}{\tan^2\mu_{p-2}}}{1+\dfrac{\sin^2\psi}{\tan^2\mu_{p-1}}};$$

from which the same general properties may be deduced, as when p is an odd number.

11. We have now demonstrated, as was proposed, the principal and leading points of this theory, for which we are indebted to M. Jacobi. For the subordinate details, and for many curious and important collateral researches that have been suggested by the new views laid open in this branch of analysis, we must refer to M. Jacobi's own work, to the papers of M. Abel, and to the writings of Legendre. We shall conclude this paper by applying the formulas that have been investigated to two particular instances, taking for p the most simple values, namely 2 and 3.

Example 1. Supposing $p = 2$.

By the formulas (3) and (6) we have these equations between the amplitudes ψ and φ, z being $= \sin \varphi$,

$$\sin \psi = \frac{\beta z \sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}}; \quad \cos \psi = \frac{1 - \frac{z^2}{\sin^2 \lambda_1}}{\sqrt{1 - k^2 z^2}}:$$

wherefore

$$1 - k^2 z^2 = \beta^2 z^2 (1 - z^2) + \left(1 - \frac{z^2}{\sin^2 \lambda_1}\right)^2:$$

and

$$\beta^2 = \frac{1}{\sin^4 \lambda_1}; \quad \frac{2}{\sin^2 \lambda_1} - \beta^2 = k^2.$$

We now get

$$k'^2 = 1 - k^2; \quad \beta = 1 + k'; \quad \sin^2 \lambda_1 = \frac{1}{1 + k'}.$$

Also, by the formula (9),

$$h = k^2 \sin^4 \lambda_1 = \frac{k^2}{(1 + k')^2} = \frac{1 - k'}{1 + k'};$$

from which we deduce

$$(1 + h)(1 + k') = 2.$$

The equation (1), viz.

$$F(h, \psi) = \beta F(k, \varphi),$$

may now be put in one or other of these two forms,

$$F(k, \varphi) = \frac{1 + h}{2} F(h, \psi),$$

$$F(h, \psi) = (1 + k') F(k, \varphi):$$

by the first we pass from the greater modulus k to the less h; and by the second, from the less modulus h to the greater k.

In the first of the two cases we must derive the amplitude ψ from φ: and, for this purpose we immediately obtain from the formula (11),

$$\tan(\psi - \varphi) = k' \tan\varphi.$$

Wherefore, if

$$k,\ h,\ h_1,\ h_2,\ \&c.$$

represent a series of decreasing moduli, of which the complements are,

$$k',\ h',\ h_1',\ h_2',\ \&c.$$

the successive quantities being derived from one another by these formulas,

$$h = \frac{1-k'}{1+k'},\quad h_1 = \frac{1-h'}{1+h'},\quad h_2 = \frac{1-h_1'}{1+h_1'},\ \&c.:$$

and, if we likewise deduce a series of amplitudes in this manner,

$$\begin{aligned}\tan(\psi - \varphi) &= k' \tan\varphi \\ \tan(\psi_1 - \psi) &= h' \tan\psi \\ \tan(\psi_2 - \psi_1) &= h_1' \tan .\ \psi_1,\ \&c.\end{aligned}$$

we shall have these successive transformations, by which the value of the given function $F(k, \varphi)$ is made to approach indefinitely to the arc of a circle,

$$F(k,\varphi) = \frac{1+h}{2} F(h, \psi)$$

$$F(k,\varphi) = \frac{1+h}{2} \cdot \frac{1+h_1}{2} F(h_1, \psi_1)$$

$$F(k,\varphi) = \frac{1+h}{2} \cdot \frac{1+h_1}{2} \cdot \frac{1+h_2}{2} \cdot F(h_2, \psi_2),\ \&c.$$

In the second case, when we would pass from the less modulus h to the greater k, the amplitude φ must be deduced from ψ. For this purpose we have

$$\frac{\sin\psi}{\cos\psi} = \frac{\beta z \sqrt{1-z^2}}{1 - \frac{z^2}{\sin^2\lambda_1}} = \frac{\beta \sin\varphi \cos\varphi}{1 - \beta \sin^2\varphi}:$$

but $\beta = \frac{2}{1+h}$; $\cos\phi\sin\phi = \frac{\sin 2\phi}{2}$; and $1 - 2\sin^2\phi = \cos 2\phi$: wherefore,

$$\frac{\sin\psi}{\cos\psi} = \frac{\sin 2\phi}{h + \cos 2\phi}; \text{ and, } \sin(2\phi - \psi) = h\sin\psi.$$

Wherefore if the quantities,

$$h, k, k_1, k_2, \&c.$$

represent a series of increasing moduli derived from one another by these equations,

$$k' = \frac{1-h}{1+h}, \qquad k_1' = \frac{1-k}{1+k}, \qquad k_2' = \frac{1-k_1}{1+k_1}, \&c.;$$

and further, if the amplitudes ψ, ϕ, ϕ_1, ϕ_2, &c. be deduced from the formulas,

$$\sin(2\phi - \psi) = h\sin\psi,$$
$$\sin(2\phi_1 - \phi) = k\sin\phi,$$
$$\sin(2\phi_2 - \phi_1) = k_1\sin\phi_1: \&c.$$

we shall have these transformations in which the successive moduli tend to the limit 1,

$$\mathrm{F}(h, \psi) = (1 + k')\,\mathrm{F}(k, \phi),$$
$$\mathrm{F}(h, \psi) = (1 + k')(1 + k_1')\,\mathrm{F}(k_1, \phi_1),$$
$$\mathrm{F}(h, \psi) = (1 + k')(1 + k_1')(1 + k_2')\,\mathrm{F}(k_2, \phi_2), \&c.$$

Example 2. Supposing $p = 3$.

By the formulas (2) and (5) we have these equations between the amplitudes ψ and ϕ,

$$\sin\psi = \frac{\beta z\left(1 - \frac{z^2}{\sin^2\lambda_2}\right)}{1 - k^2 z^2 \sin^2\lambda_2}; \quad \cos\psi = \sqrt{1 - z^2} \cdot \frac{1 - \frac{z^2}{\sin^2\lambda_1}}{1 - k^2 z^2 \sin^2\lambda_2}:$$

wherefore,

$$(1 - k^2 z^2 \sin^2\lambda_2)^2 = \beta^2 z^2 \left(1 - \frac{z^2}{\sin^2\lambda_2}\right)^2 + (1 - z^2)\left(1 - \frac{z^2}{\sin^2\lambda_1}\right)^2:$$

from which we get

$$2\,k^2 \sin^2\lambda_2 = \frac{2}{\sin^2\lambda_1} + 1 - \beta^2,$$

$$k^4 \sin^4 \lambda_2 = \frac{1}{\sin^4 \lambda_1} + \frac{2}{\sin^2 \lambda_1} - \frac{2\beta^2}{\sin^2 \lambda_2},$$

$$\beta = \frac{\sin^2 \lambda_2}{\sin^2 \lambda_1}.$$

Now, observing that $\frac{\beta^2}{\sin^2 \lambda_2} = \frac{\beta}{\sin^2 \lambda_1}$, we obtain by equating the values of $k^4 \sin^4 \lambda_2$,

$$\left(\frac{1}{\sin^2 \lambda_1} + \frac{1-\beta^2}{2}\right)^2 = \frac{1}{\sin^4 \lambda_1} + \frac{2(1-\beta)}{\sin^2 \lambda_1}:$$

from which we deduce

$$\beta = \frac{2}{\sin \lambda_1} - 1.$$

In order to simplify the formulas I shall put $\frac{1}{\sin \lambda_1} = 1 + \varepsilon$: then $\beta = 1 + 2\varepsilon$, and $\sin^2 \lambda_2 = \beta \sin^2 \lambda_1 = \frac{1 + 2\varepsilon}{(1+\varepsilon)^2}$: and, having substituted these values in the first of the foregoing equations, we shall get,

$$\varepsilon^4 + 2\varepsilon^3 - 2k'^2 \varepsilon - k'^2 = 0.$$

This equation may be resolved by the usual method into the two following quadratic factors,

$$\varrho^3 = 4k^2 k'^2,$$

$$\varepsilon^2 + (1 + \sqrt{1-\varrho})\,\varepsilon - \tfrac{1}{2}\sqrt{\varrho^2 + 4k'^2} - \tfrac{1}{2}\varrho = 0,$$

$$\varepsilon^2 + (1 - \sqrt{1-\varrho})\,\varepsilon + \tfrac{1}{2}\sqrt{\varrho^2 + 4k'^2} - \tfrac{1}{2}\varrho = 0.$$

The second of these equations has two impossible roots: the first has two real roots, one being negative and foreign to the question, and the other positive, which solves the problem. Thus ε has only one value, which may be constructed geometrically, but the algebraic expression of it need not be written down.

We have next to derive the amplitude ψ from φ. We readily obtain from the foregoing biquadratic equation,

$$k^2 = \frac{(1+\varepsilon)^3(1-\varepsilon)}{1+2\varepsilon};$$

and hence,

$$1 - k^2 \sin^2 \lambda_2 = c_2^2 = \varepsilon^2.$$

And the formula (10) will determine ψ when ϕ is given.

Finally, therefore, we have these determinations,

$$\beta = 1 + 2\,\varepsilon,\ h = \frac{k^3}{(1+\varepsilon)^4},\ \tan\left(\frac{\psi - \phi}{2}\right) = \varepsilon \tan \phi\,;$$

$$\mathrm{F}\,(k, \phi) = \frac{1}{1 + 2\,\varepsilon} \times \mathrm{F}\,(h, \psi),$$

the modulus h being less than k^3. By repeating the like operations, a succession of moduli rapidly decreasing may be formed, by means of which the given elliptic function will be reduced to a circular arc as near as may be required.